

Leapover lengths and first passage time statistics for Lévy flights

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Exact results for the first passage time and leapover statistics of symmetric and one-sided Lévy flights (LFs) are derived. LFs with stable index α are shown to have leapover lengths, that are asymptotically power-law distributed with index α for one-sided LFs and, surprisingly, with index $\alpha/2$ for symmetric LFs. The first passage time distribution scales like a power-law with index $1/2$ as required by the Sparre Andersen theorem for symmetric LFs, whereas one-sided LFs have a narrow distribution of first passage times. The exact analytic results are confirmed by extensive simulations.

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The statistics of first passage times is a classical concept to quantify processes, in which it is of interest when the dynamic variable crosses a given threshold value for the first time, e.g., when a tracer in some aquifer reaches a certain probe position, two molecules meet to form a chemical bond, animals search for sparse food locations, or a share at the stock market crosses a preset market value [1, 2, 3]. Here, we revisit the first passage time problem for processes with non-trivial jump length distributions, namely, Lévy flights (LFs) and derive exact asymptotic expressions for the first passage time density $p_f(\tau)$ of symmetric and one-sided LFs. For the former, we obtain the Sparre Andersen universality $p_f(\tau) \simeq \tau^{-3/2}$, while a narrow behavior is found for one-sided LFs. Apart from calculating the first passage times, we investigate the behavior of the first passage leapovers, that is, the distance the random walker overshoots the threshold value d in a single jump (see Fig. 1). Surprisingly, for symmetric LFs with jump length distribution $\lambda(x) \simeq |x|^{-1-\alpha}$ ($0 < \alpha < 2$) the distribution of leapover lengths across $x = d$ is distributed like $p_l(\ell) \simeq \ell^{-1-\alpha/2}$, i.e., it is much broader than the original jump length distribution. In contrast, for one-sided LFs the scaling of $p_l(\ell)$ bears the same index α .

For processes subject to a narrow jump length distribution with finite second moment $\int_{-\infty}^{\infty} x^2 \lambda(x) dx$ the crossing of a given threshold value d is identical to the first arrival at $x = d$ [2]. This is no longer true for LFs: Intuitively, a particle, whose jump lengths are distributed according to the symmetric long-tailed distribution $\lambda(x) \simeq |x|^{-1-\alpha}$ ($0 < \alpha < 2$) is likely to criss-cross the point $x = d$ multiple times before it eventually hits it, causing the first arrival at d to be slower than its first passage across d [4]. A measure for the ability to criss-cross d is the distribution of leapover lengths, $p_l(\ell)$. Information on the leapover behavior is therefore important to the understanding of how far proteins searching for their specific binding site along DNA overshoot their target [5], climatic forcing visible in ice core records exceeds a given value [6], or to define better stock market strategies determining when to buy or sell a certain share

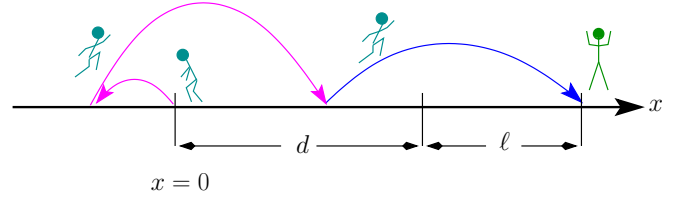


Figure 1: Schematic of the leapover problem: the random walker starts at $x = 0$ and after a number of jumps crosses the point $x = d$, overshooting it by a distance ℓ .

instead of fixing a threshold price [7]. The quantification of leapovers is vital to estimate how far diseases would spread once a carrier of that disease crosses a certain border [8]. Leapover statistics of one-sided LFs provide an interesting alternative interpretation of the distribution of the first waiting time in ageing continuous time random walks [9], just to name a few examples.

The master equation for a Markovian diffusion process,

$$\frac{\partial P(x, t)}{\partial t} = \frac{1}{\tau} \int_{-\infty}^{\infty} [\lambda(x - x') P(x', t) - \lambda(x' - x) P(x, t)] dx' \quad (1)$$

accounts for the influx of probability to position x , and the outflux away from x , where $\lambda(x)$ is a general, normalized jump length distribution. The time scale for single jumps is τ . The solution to Eq. (1) in Fourier space is $P(k, t) = e^{-[1 - \lambda(k)]t/\tau}$, denoting the Fourier transform $f(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) dx$ by explicit dependence on the wave number k . For instance, for the symmetric jump length distribution $\lambda(x) \simeq \sigma^\alpha |x|^{-1-\alpha}$, one finds

$$P(k, t) = e^{-K^{(\alpha)} |k|^\alpha t} \quad (2)$$

with $K^{(\alpha)} = \sigma^\alpha / \tau$, the characteristic function of a symmetric Lévy stable law as obtained from continuous time random walk theory in the diffusion limit or from the equivalent space fractional diffusion equation [10].

In the following we study processes with the long-tailed composite jump length distribution

$$\lambda(x)/\tau = \Theta(|x| - \varepsilon) [c_1 \Theta(-x) + c_2 \Theta(x)] / |x|^{1+\alpha}, \quad (3)$$

where $\Theta(x)$ is the Heaviside function. For $c_1 = c_2$, $\lambda(x)$ defines a symmetric LF, and for $c_1 = 0$ and $c_2 > 0$ a completely asymmetric (one-sided) LF permitting exclusively forward jumps. The cutoff ε excludes the singularity at $x = 0$, but can be taken to be small, $\varepsilon \rightarrow 0$ [11].

In the theory of homogeneous random processes with independent jumps there exists a theorem, which provides an exact expression for the joint PDF $p(\tau, \ell)$ of first passage time τ and leapover length ℓ ($\ell \geq 0$) across $x = d$ for a particle initially seeded at $x = 0$ [12, 13]. We here evaluate this theorem, that appears to have been widely overlooked, and derive a number of new analytic results for $p_f(\tau)$ and $p_l(\ell)$ of symmetric and one-sided LFs. With the probability to jump longer than x ,

$$\mathcal{M}(x) = \int_x^\infty \lambda(x') dx', \quad x > 0, \quad (4)$$

the theorem states that the double Laplace transform of the joint PDF [12, 13]

$$p(u, \mu) = \int_0^\infty \int_0^\infty e^{-u\tau - \mu\ell} p(\tau, \ell) d\tau d\ell \quad (5)$$

is given in terms of the multiple integral

$$\begin{aligned} p(u, \mu) &= 1 - q_+(u, d) - \frac{\mu}{u} \int_0^d \frac{\partial q_+(u, s)}{\partial s} ds \\ &\times \int_{-\infty}^0 \frac{\partial q_-(u, s')}{\partial s'} ds' \int_0^\infty e^{-\mu s''} \\ &\times \mathcal{M}(d + s'' - s' - s) ds''. \end{aligned} \quad (6)$$

Here, we use the two auxiliary measures $q_\pm(u, x)$ defined through Fourier transforms

$$\begin{aligned} \tilde{q}_\pm(u, k) &= \int_{-\infty}^\infty e^{ikx} \frac{\partial q_\pm(u, x)}{\partial x} dx \\ &= \exp \left\{ \pm \int_0^\infty \frac{e^{-ut}}{t} \int_0^{\pm\infty} (e^{ikx} - 1) P(x, t) dx dt \right\}, \end{aligned} \quad (7)$$

and the condition $q_\pm(u, 0) = 0$. They are related to the cumulative distributions of the maximum, $Q_+(t, d) = \Pr\{\max_{0 \leq \tau \leq t} x(\tau) < d\}$, and minimum, $Q_-(t, d) = \Pr\{\min_{0 \leq \tau \leq t} x(\tau) < d\}$, of the position $x(t)$ such that $q_\pm(u, d) = u \int_0^\infty e^{-ut} Q_\pm(t, d) dt$. The complicated integrals above reduce to elegant results for symmetric and one-sided LFs, as we show now.

For *symmetric LFs* ($c_1 = c_2 \equiv c$), the propagator is defined by the characteristic function (2) with generalized diffusion coefficient $K^{(\alpha)} = 2c\Gamma(1 - \alpha)\cos(\pi\alpha/2)/\alpha$. In the limit $u \rightarrow 0$ (long time limit), we obtain from Eq. (7)

$$\tilde{q}_+(u, k) \sim \frac{u^{1/2}}{\sqrt{K^{(\alpha)}}|k|^{\alpha/2}} \exp \left\{ \frac{i \operatorname{sign}(k) \pi \alpha}{4} \right\}. \quad (8)$$

Inverse Fourier transform yields

$$q_+(u, d) \sim \frac{2u^{1/2}}{\alpha\sqrt{K^{(\alpha)}}\Gamma(\alpha/2)} d^{\alpha/2}, \quad d > 0 \quad (9)$$

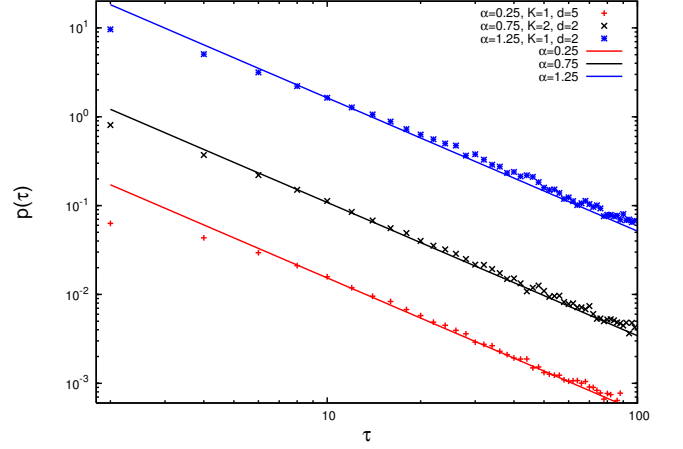


Figure 2: First passage time density $p_f(\tau)$ for symmetric LFs with Sparre Andersen universality $p_f(\tau) \simeq \tau^{-3/2}$. The curves for $\alpha = 0.75$ and 1.25 are multiplied by a factor 10 and 100, for better visibility. Lines: theory. Symbols: simulations.

such that from $p_f(u) = 1 - q_+(u, d)$ we find

$$p_f(\tau) \sim \frac{d^{\alpha/2}}{\alpha\sqrt{\pi K^{(\alpha)}}\Gamma(\alpha/2)} \tau^{-3/2}. \quad (10)$$

This is the exact asymptotic first passage time PDF of symmetric LFs. Fig. 2 demonstrates good agreement with simulations results, for which the algorithm from Ref. [14] was used to obtain random numbers distributed according to Lévy stable laws. We note that previously only the $\tau^{-3/2}$ scaling was known from simulations and application of the Sparre Andersen theorem [4].

For symmetric LFs, for $0 < \alpha < 2$ we obtain that

$$\mathcal{M}(x) = \frac{K^{(\alpha)}}{2\Gamma(1 - \alpha)\cos(\pi\alpha/2)} x^{-\alpha}, \quad x > 0. \quad (11)$$

Using that for symmetric LFs $q_-(\tau, x) = q_+(\tau, -x)$ it turns out after some transformations from Eq. (6) that

$$p_l(\mu) = \int_0^\infty e^{-\mu\ell} \frac{\sin(\pi\alpha/2)}{\pi} \frac{(d/\ell)^{\alpha/2}}{d + \ell} d\ell, \quad (12)$$

from which it follows immediately that

$$p_l(\ell) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{d^{\alpha/2}}{\ell^{\alpha/2}(d + \ell)}, \quad (13)$$

see Fig. 3. Note that p_l is normalized. In the limit $\alpha \rightarrow 2$, $p_l(\ell)$ tends to zero if $\ell \neq 0$ and to infinity at $\ell = 0$ corresponding to the absence of leapovers in the Gaussian continuum limit. However, for $0 < \alpha < 2$ the leapover PDF follows an asymptotic power-law with index $\alpha/2$, and is thus broader than the original jump length PDF $\lambda(x)$ with index α . This is a remarkable finding: while λ for $1 < \alpha < 2$ has a finite characteristic length $\langle |x| \rangle$, the mean leapover length diverges.

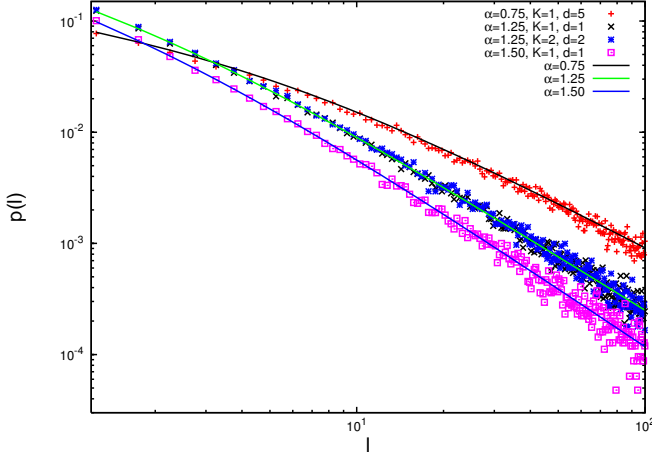


Figure 3: Leapover density $p_l(\ell)$ for symmetric LFs.

Consider now *one-sided* LFs with $c_1 = 0$ in Eq. (3). In this case, the PDF has the characteristic function

$$P(k, t) = \exp \left\{ -K^{(\alpha)} t |k|^\alpha \left[1 - i \operatorname{sign}(k) \tan \left(\frac{\pi\alpha}{2} \right) \right] \right\}, \quad (14)$$

where $K^{(\alpha)} = c_2 \Gamma(1-\alpha) \cos(\pi\alpha/2)/\alpha$ and $\mathcal{M}(x)$ for $x > 0$ is twice the expression in Eq. (11). Eq. (7) leads to

$$\tilde{q}_+(u, k) = \frac{u}{u + \zeta}, \quad \zeta = K^{(\alpha)} (-ik)^\alpha / \cos \left(\frac{\pi\alpha}{2} \right), \quad (15)$$

as $(-ik)^\alpha = [-i \operatorname{sign}(k) |k|]^\alpha = |k|^\alpha \exp[-i \operatorname{sign}(k) \pi\alpha/2]$. From this we calculate that the Fourier transform of $\langle \exp(-u\tau) \rangle = \int_0^\infty \exp(-u\tau) p_f(\tau) d\tau$ can be written as

$$\int_{-\infty}^\infty e^{ikx} \langle e^{-u\tau} \rangle dx = \frac{(-ik)^{\alpha-1}}{(-ik)^\alpha + u \cos(\pi\alpha/2)/K^{(\alpha)}}, \quad (16)$$

and we change the variable $ik \rightarrow -s$ to find [15]

$$\langle e^{-u\tau} \rangle = E_\alpha \left[-\frac{u}{K^{(\alpha)}} \cos \left(\frac{\pi\alpha}{2} \right) d^\alpha \right]. \quad (17)$$

Here, we used the definition of the Mittag-Leffler function

$$\int_0^\infty E_\alpha(-\theta x^\alpha) e^{-sx} dx = \frac{s^{\alpha-1}}{s^\alpha + \theta}. \quad (18)$$

whose series expansion and asymptotic behavior are [10]

$$E_\alpha(-z) = \sum_{n=0}^\infty \frac{(-z)^n}{\Gamma(1+\alpha n)} \sim \sum_{n=0}^\infty \frac{(-1)^n z^{-1-n}}{\Gamma(1-\alpha[n+1])}. \quad (19)$$

From the relation between E_α and the M_α -function [16],

$$\int_0^\infty e^{-ut} M_\alpha(t) dt = E_\alpha(-u), \quad 0 < \alpha < 1, \quad (20)$$

the following result for the first passage time PDF yields

$$p_f(\tau) = \frac{K^{(\alpha)}}{\cos(\alpha\pi/2) d^\alpha} M_\alpha \left(\frac{K^{(\alpha)} \tau}{\cos(\alpha\pi/2) d^\alpha} \right). \quad (21)$$

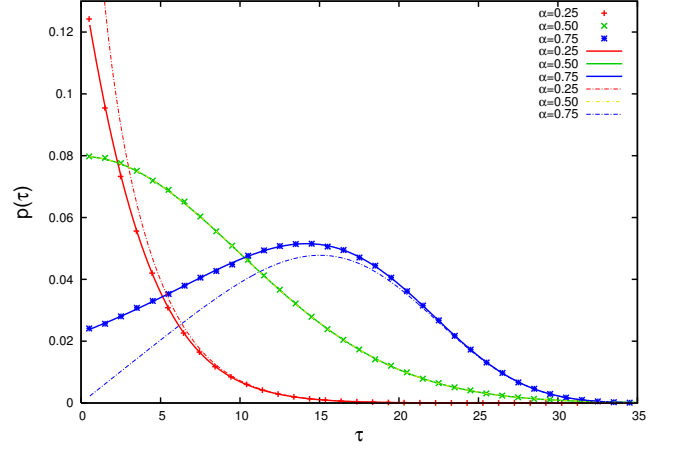


Figure 4: First passage density for one-sided LF ($K^{(\alpha)} = 1$). The thick lines represent numerical evaluations of the exact analytic expression, while the thin dashed lines correspond to the asymptotic behavior (23). Symbols: simulations.

The M_α -function has the series representation and asymptotic behavior with exponential decay

$$M_\alpha(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n! \Gamma(1-\alpha-\alpha n)} \quad (22)$$

$$\sim \frac{(\alpha z)^{(\alpha-1/2)/(1-\alpha)}}{\sqrt{2\pi(1-\alpha)}} \exp \left[-\frac{1-\alpha}{\alpha} (\alpha z)^{1/(1-\alpha)} \right]. \quad (23)$$

The moments of the M_α -function are obtained through

$$\int_0^\infty z^n M_\alpha(z) dz = \lim_{s \rightarrow 0} (-1)^n \frac{d^n}{ds^n} E_\alpha(-s) = \frac{\Gamma(n+1)}{\Gamma(1+\alpha n)}, \quad (24)$$

from which we calculate the mean first passage time

$$\langle \tau \rangle = \frac{d^\alpha \cos(\pi\alpha/2)}{K^{(\alpha)} \Gamma(1+\alpha)}, \quad (25)$$

that is *finite* and grows with the α th power of the distance d . For $\alpha = 1/2$, we recover the exact form [17]

$$p_f(\tau) = K^{(\alpha)} \sqrt{\frac{2}{\pi d}} \exp \left(-\frac{(K^{(\alpha)})^2 \tau^2}{2d} \right). \quad (26)$$

The first passage PDF $p_f(\tau)$ is displayed in Fig. 4 in nice agreement with the simulations. Note that for $\alpha \leq 1/2$ the tail of $\lambda(x)$ is so long that it is most likely to cross $x = d$ in the first jump, while for $\alpha > 1/2$, $p_f(\tau)$ has a maximum at finite $\tau > 0$.

To obtain the leapover statistics for the one-sided LF, we first note that since $P(x < 0, t) = 0$ (only forward steps are permitted) we have $q_-(u, k) = 1$, and thus $\partial q_-(u, x)/\partial x = \delta(x)$. Combining Eqs. (6) and (7),

$$\langle e^{-\mu\ell} \rangle = 1 - \lim_{u \rightarrow 0} \frac{\mu}{u} \int_0^d \int_0^\infty e^{-\mu s'} \mathcal{M}(d+s'-s) \times \frac{\partial q_+(u, s)}{\partial s} ds' ds. \quad (27)$$

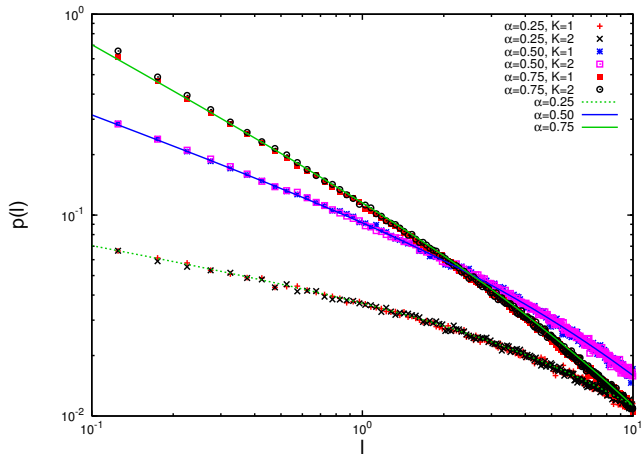


Figure 5: Leapover distribution for one-sided LF with $d = 10$.

With the small u expansion of the Mittag-Leffler function, Eqs. (17) and (19) produce

$$\frac{\partial q_+(u, x)}{\partial x} = \frac{u \cos(\pi\alpha/2)}{K^{(\alpha)}\Gamma(\alpha)} x^{\alpha-1}. \quad (28)$$

Eqs. (15) and (28) inserted into Eq. (27) then yield

$$p_l(\mu) = \langle e^{-\mu\ell} \rangle = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty e^{-\mu\ell} \frac{d^\alpha}{\ell^\alpha(d+\ell)}, \quad (29)$$

leading to the leapover PDF

$$p_l(\ell) = \frac{\sin(\pi\alpha)}{\pi} \frac{d^\alpha}{\ell^\alpha(d+\ell)}, \quad (30)$$

which corresponds to the result obtained in Ref. [17] from a different method. Thus, for the one-sided LF, the scaling of the leapover is exactly the same as for the jump length distribution, namely, with exponent α .

The leapover distribution (30) also provides a new aspect to the first waiting time in a renewal process with broad waiting time distribution $\psi(t) \simeq t^{-1-\beta}$ ($0 < \beta < 1$). Interpret the position x as time and the jump lengths drawn from the one-sided $\lambda(x)$ as waiting times t . Consider an experiment, starting at time t_0 , on a system prepared at time 0 (corresponding to position $x = 0$). Then the first recorded waiting time t_1 of the system will be distributed like $p_1(t_1) = \pi^{-1} \sin(\pi\alpha) t_0^\alpha / [t_1^\alpha (t_0 + t_1)]$, as obtained from a different reasoning in Ref. [9]. We note that the first passage time τ in this analogy corresponds to the number of waiting events.

While for symmetric LFs it was previously established that the first passage time distribution follows the universal Sparre Andersen asymptotics $p_f(\tau) \simeq \tau^{-3/2}$, here we derived the prefactor of this law, in particular, its dependence on the generalized diffusion coefficient $K^{(\alpha)}$.

For the same case, we derived the leapover distribution $p_l(\ell)$, that is surprising for two reasons: first, $p_l(\ell)$ is independent of $K^{(\alpha)}$, synonymous to the noise strength; and second, its power-law exponent is $\alpha/2$, and thus $p_l(\ell)$ is broader than the original jump length distribution.

For one-sided LFs, we recovered the previously reported leapover distribution and derived the so far unknown first passage time distribution. While the leapovers follow the same asymptotic scaling $p_l(\ell) \simeq \ell^{-1-\alpha}$ as the jump lengths $\lambda(x)$, once more independent of $K^{(\alpha)}$, the first passage times are narrowly distributed. We also drew an analogy between the leapovers and the first waiting time in a subdiffusive renewal process. For both symmetric and one-sided LFs, extensive simulations showed nice agreement with the theoretical results, without adjustable parameters.

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